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Turbulence in a gas laser

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ABSTRACT

We test a recent assertion [A. Muriel, Physica A 388 (4) (2009) 311] that a gas consisting of excited molecules is turbulent, in contrast to the laminar state of a gas of ground state molecules. Since a lasing gas is made up of excited molecules, we examine if a lasing gas system is indeed turbulent. Surprisingly, from a literature search, it appears that turbulence in a lasing gas medium has never been addressed. To test for turbulence, we use a recently proposed criterion for the existence of turbulence, the presence of multivalued steady-state velocity fields [P. Getreur, A. Albano, A. Muriel, Phys. Lett. A 366 (2007) 101]. To study this subject, we improve an old model of a gas of two-level atoms in a one-dimensional model [A. Muriel, M. Dresden, Physica D 94 (1996) 103] by including the effect of a radiation field with the use of Einstein A and B coefficients. A set of coupled equations for the velocity fields in one dimension are derived. The zeroth order implementation of an iterative solution establishes that the steady-state velocity fields are multivalued, given by the Lambert function. We obtain signature characteristics of turbulence such as velocity reversals, infinite gradients, and stagnation points.

There have been a series of studies [1–6] which hinted turbulent-like behavior in a gas of two-level atoms. The studies modeled the transitions between the population of particles in the two energy levels by means of radiation emission. The radiated photons were no longer taken into account once they were emitted. The results of this approach hinted at turbulent-like behavior.

In this letter, we improve the above models by extending the analysis to include the effect of the radiation field arising from the two-state transitions. The emitted photons are now accounted for by considering not only spontaneous but also stimulated emission and absorption processes. As a result, we find turbulent behavior as defined in Ref. [7], instead of old imprecise hints.

Consider a two-level system whose time evolution is determined by a radiation field of density ρ. Let $f_1(x, v, t)$ and $f_2(x, v, t)$ be the classical distribution functions for particles in the ground energy state and the excited energy state, respectively. We replace the collision term in the Boltzmann transport equation with transition terms due to the interaction with the radiation bath. The resulting coupled equations are:

$$\frac{\partial f_1}{\partial t} + \frac{p}{m}\frac{\partial f_1}{\partial x} = Af_2 + \rho B_{21} f_2 - \rho B_{12} f_1 \tag{1}$$

$$\frac{\partial f_2}{\partial t} + \frac{p}{m}\frac{\partial f_2}{\partial x} = -Af_2 - \rho B_{21} f_2 + \rho B_{12} f_1 \tag{2}$$

where m is the mass of a gas particle and p is the momentum. The rate of spontaneous emission is given by A, while the rates for stimulated emission and absorption are given by ρB_{21} and ρB_{12}, respectively. We let the integral $\int \mathrm{d}v v$ act on both

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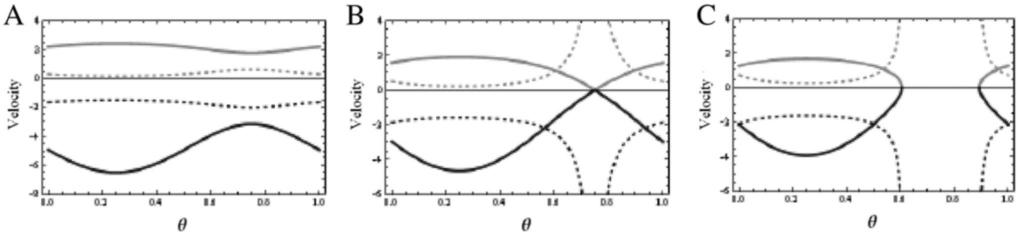

Fig. 1. The solid lines represent the velocity, while the dashed lines are the rate of change of velocity with respect to θ. The lines correspond to the $k = 0$ and $k = -1$ branches of the Lambert function, respectively. For all plots, $|\mu^2|$, L, ρ and C_1 are set to unity, while $v_0 = 2.12$. The frequency ω_0 for A, B, C are respectively 1.5, 1, 1/9.

sides of these equations and identify $v_1 = \int \mathrm{d}v v f_1$ and $v_2 = \int \mathrm{d}v v f_2$. We then use the standard coarse-graining approach to derive the following pair of Burgers-like equations [4]:

$$\frac{\partial v_1}{\partial t} + \frac{1}{2}\frac{\partial (v_1^2)}{\partial x} = Av_2 + \rho B v_2 - \rho B v_1 \tag{3}$$

$$\frac{\partial v_2}{\partial t} + \frac{1}{2}\frac{\partial (v_2^2)}{\partial x} = Av_1 + \rho B v_1 - \rho B v_2 \tag{4}$$

where we have assumed that the rates of stimulated emissions obey $B_{12} = B_{21} = B$. Once the rationale for the above derivation is established, it would in fact be simpler to postulate the existence of the above equations and study their solution.

Let us focus on the steady-state solutions of the above pair:

$$\frac{1}{2}\frac{\partial (v_1^2)}{\partial x} = Av_2 + \rho B v_2 - \rho B v_1 \tag{5}$$

$$\frac{1}{2}\frac{\partial (v_2^2)}{\partial x} = Av_1 + \rho B v_1 - \rho B v_2 . \tag{6}$$

We solve this pair of equations by iteration. First assume that the velocity of the atoms in the excited state is constant, $v_2 = v_{20} > 0$, and we find that Eq. (5) becomes

$$\frac{1}{2}\frac{\partial (v_2^2)}{\partial x} = (A + \rho B)v_{20} - \rho B v_1 . \tag{7}$$

First we define

$$F = 1 + \frac{(x+c)\rho^2 B^2}{v_{20}(A+\rho B)} \tag{8}$$

and express the solution for the first iteration of $v_1(x)$:

$$v_1(x) = -\rho B(x+c) - (A+B) - (A+B)F + 2v_{20}\, LambertW\left(\frac{\mathrm{e}^{-F}}{(A+\rho B)v_{20}}\right)\Big/\rho \tag{9}$$

where c is a constant.

Substituting this on the rhs of Eq. (5) to find the first iteration of $v_2(x)$ we get

$$v_2(x) = \mp\left[2(A+\rho B)\int v_1(x)\mathrm{d}x - 2\rho B + d\right]^{1/2} \tag{10}$$

and d is another constant. $v_2(x)$ is multivalued in two ways, the square root, and implicitly from the Lambert solution of $v_1(x)$.

We stop here because we have already established the multivalued aspect of the velocity fields.

We now analyze $v_1(x)$ to illustrate some properties of the velocity field.

The rate of spontaneous decay A and the rate of stimulated process B are related, $A = \frac{\omega^3 h}{2\pi c^3}B$. By Fermi's Golden Rule, $B = \frac{\pi}{3\epsilon_0 \hbar^2}|\mu|^2$ where μ^2 is the probability for the transitions. The energy gap between the excited state and the ground state is $h\omega/2\pi$.

We impose periodic boundary conditions by choosing a circular geometry, setting $x = L\sin(2\pi\theta)$. Fig. 1 shows three representative velocity solutions for different parameter sets on the interval $0 \le \theta \le 2\pi$. We have adopted units where c, $\frac{h}{2\pi}$, ε_0 are unity.

Plots A, B, C show the resulting bifurcation in velocity. The bifurcation corresponds to the real valued $z = 0$ and $z = -1$ branches of the Lambert function. Note that the bifurcation includes a negative and a positive branch allowing the velocity to reverse, a characteristic signature of turbulence [8]. The gap between these two flows increases with the frequency ω_0.

The bifurcation in plot A is characterized by velocity curves whose change in angular position θ is always finite. $\frac{\mathrm{d}v}{\mathrm{d}\theta}$ is the dashed line. On the other hand the velocity curves in plots B and C include infinite derivatives at certain points. For plot B this happens at the cusps where both branches meet at $v_1 = 0$. These cusps are interesting because they indicate that there are flows in the ground state with zero velocity, possibly an example of the so-called stagnation points [9] as in real turbulence. Meanwhile for plot C the rates of change of velocity become infinite, approaching regions where the solutions cease to exist. In such regions the solutions for both branches become imaginary.

We comment that these infinite rates of change of velocity with respect to angular position are suggestive of a mathematical criterion suggested by Ruelle [10] that turbulence begins when the gradients become infinite.

In principle, one may further iterate the solutions using the self-consistency of the system given by Eqs. (5) and (6). This is work for the future, at this time, we start at the zeroth order approximation with $v_2 = v_{20}$ (5), substitute v_1 into Eq. (6) to obtain the next iteration of v_2 from Eq. (6), and stop there. The results are already interesting in showing multivaluedness of the velocity field, our objective in this letter.

What makes us conclude that we have shown turbulent behavior? In the first place, there has not been a unique universally accepted theoretical criterion for turbulence. We adopt a new one [7,11], from which we have made some conclusions: any transport equation with exact steady-state solutions which are multivalued, displaying multivalued velocity fields, can be used to model turbulence. One must simply provide a physical mechanism for transitions from one allowed solution to another. In principle, such a mechanism maybe found from electrical, thermal or mechanical perturbation. Clearly this is yet to be elucidated. But this approach requiring the existence of multivalued steady states from which transitions could occur is simpler than a brute-force attack on time dependent transport equations like the Navier–Stokes equation. Instead, we could adopt post-Navier–Stokes equations with new physical content that easily produces what we define in Refs. [7,11] as turbulence, perhaps the most precise definition to date. That radiation is important is not surprising, all turbulent systems eventually heat up. That quantum concepts are necessary to demonstrate turbulence in our model is new. Finally, since our literature search has not yielded any work on the turbulence of the lasing gas itself, it is suggested that experiments be designed to detect turbulence in the lasing gas medium. This will be an important test of all the new ideas quoted in this letter: from the definition of turbulence, to the turbulent excited gas [11], and implicitly, the quantum origin of turbulence [7] for the lasing medium.

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